ABSTRACT

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The classical Schoenflies Theorem states that for any Jordan curve J in the Euclidean plane E^2, there exists a homeomorphism h of E^2 onto itself such that $h(J) = S^1$. The generalized Schoenflies Theorem states that if h is a homeomorphic embedding of $S^{n-1} \times [0,1]$ into the standard n-sphere S^n, then the closure of either complementary domain of $h(S^{n-1} \times \{1/2\})$ is a topological n-cell. In this thesis, we will show that for any Jordan curve J in E^2 there exists a homeomorphic embedding $h:S^1 \times [0,1]$ into E^2 such that $h(S^1 \times \{1/2\} = J$, thereby showing that the classical Schoenflies Theorem is a consequence of the generalized Schoenflies Theorem.

THE CLASSICAL AND GENERALIZED

SCHOENFLIES THEOREMS

by

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INTRODUCTION

A Jordan curve in the Euclidean plane E^2 is any subset of the plane which is homeomorphic to the unit circle S^1. The Jordan Curve Theorem states that if J is a Jordan curve in E^2, then $E^2 - J$ is the union of exactly two mutually exclusive domains, one bounded and one unbounded, denoted J_B and J_U, respectively; furthermore J is the frontier of each of J_B and J_U. The theorem was first stated by Camille Jordan in 1866, and the first valid proof, published in 1905, was given by Oswald Veblen.

A more interesting result is the classical Schoenflies Theorem which states that for any Jordan curve J in E^2, there exists a homeomorphism h of E^2 onto itself such that $h(J) = S^1$ [9]. See Newman [8] for a proof. Note that the Jordan Curve Theorem is an immediate corollary of the Schoenflies Theorem. For an $(n-1)$ - sphere in n-dimensional Euclidean space E^n with $n \geq 3$ the result fails because of the existence of "wild" embeddings. See for example the classic paper of Fox and Artin [3].

Given a homeomorphic embedding h of $S^{n-1} \times [0,1]$ in S^n (sometimes referred to as the "collaring hypothesis" on $h(S^{n-1} \times \{1/2\})$), the Generalized Schoenflies Problem asks whether the closures of the complementary domains of $h(S^{n-1} \times \{1/2\})$ in S^n are topological n-cells. (By topological n-cell we mean any space homeomorphic to the standard n-cell $\{x \in E^n \mid | x | \leq 1\}$.) Barry Mazur [6] gave an affirmative answer with a certain "niceness" condition on h. Marston

Morse [7] showed that Mazur's theorem implied the theorem without the "niceness" condition. Independently, Morton Brown [2] proved the theorem with no extra conditions on h required.

In this thesis we will show that every Jordan curve in E^2 satisfies the collaring hypothesis of the generalized Schoenflies Theorem. As a result, we obtain the classical Schoenflies Theorem.

In Chapter I, certain definitions and results regarding the first steps of the proof of the Schoenflies Theorem are considered.

In Chapter II, a survey of Brown's proof of the generalized Schoenflies Theorem is given.

In Chapter III, we show that every Jordan curve in E^2 satisfies the collaring hypothesis.

A working knowledge of elementary topology is assumed. The reader is referred to [1], [4], [5], [8], and [10] for definitions and theorems not covered in this thesis. For any two spaces X and Y, a continuous function $f: X \to Y$ will be referred to as a <u>map</u> from X to Y. If $A \subset X$, then the <u>closure</u> of A will be denoted $Cl(A)$; the <u>diameter</u> of A by diam A; the <u>frontier</u> of A by $Fr(A)$; and the <u>interior</u> of A by $Int(A)$. Also, the closed interval of real numbers $[0,1]$ will be denoted by I. The set of positive integers will be denoted by Z^+, and E^1 will be used to denote the set of real numbers. If (X,d) is a metric space, $x \in X$, and $\epsilon > 0$, then $U(x,\epsilon) = \{y \in X \mid d(x,y) < \epsilon\}$. We let E^n be $\{(x_1, x_2, \cdots, x_n) \mid x_i \in E^1 \text{ if } i = 1, 2, \cdots, n\}$ so that E^n is a metric space where

$d(x,y) = |x-y| = (\Sigma_{i=1}^{n}(x_i-y_i)^2)^{1/2}$ if $x = (x_1, x_2, \cdots, x_n) \in E^n$ and $y \in (y_1, y_2, \cdots, y_n) \in E^n$. We define the map $e: E^1 \to S^1$ by $e(x) = \exp(2\pi ix)$ for all $x \in E^1$.

CHAPTER I

ACCESSIBILITY AND THE POLYHEDRAL SCHOENFLIES THEOREM

The following lemma is a criterion for continuity which will be used repeatedly throughout this thesis.

LEMMA 1.1 (GLUING LEMMA) Let X and Y be spaces. If $X = A \cup B$ where A and B are closed and $f\colon X \to Y$ is a function such that $f|A$ and $f|B$ are continuous, then f is continuous.

PROOF: Let C be a closed subset of Y. Since $f|A$ is continuous, $(f|A)^{-1}(C)$ is closed in A, and hence in X. Similarly, $(f|B)^{-1}(C)$ is closed in X. Since $A \cup B = X$, $f^{-1}(C) = (f|A)^{-1}(C) \cup (f|B)^{-1}(C)$, which is closed in X. Therefore, f is continuous. $\square$

DEFINITION 1.2: A space X is path connected if for any two points $x,y \in X$ there exists a map $f\colon I \to X$ such that $f(0) = x$ and $f(1) = y$. In general, a map $f\colon I \to X$ is called a path in X, and we say that the path above joins x to y.

DEFINITION 1.3: Any space homeomorphic to I is called an arc. (We also sometimes refer to a singleton set as a degenerate arc.)

THEOREM 1.4: Let A,B be compact subsets of E^2; a,b points of $E^2 - (A \cup B)$. If neither A nor B separates a from b, and if $A \cap B$ is connected (or empty), then $A \cup B$ does not separate a from b.

The above theorem is a consequence of the Mayer–Vietoris Theorem and Eilenberg's criterion. See Wall [10] for a proof. □

DEFINITION 1.5: Any space homeomorphic to the union of a circle and one of its diameters is a theta curve.

LEMMA 1.6: Given $x \in J$ where $J \subset E^2$ is a Jordan curve. Then for all positive $\epsilon < \text{diam } J$, there exists an arc K in J such that $x \in K$ and $K \cup Fr(U(x,\epsilon))$ is a theta curve.

PROOF: Let $x \in J$, $\epsilon < \text{diam } J$, and $y \in J - Cl(U(x,\epsilon))$. Since $Cl(U(x,\epsilon))$ and $\{y\}$ are disjoint compact sets, $\delta = d(y, Cl(U(x,\epsilon))) > 0$. Since J is locally path connected, the neighborhood $U = J \cap U(y, \frac{\delta}{2})$ of y in J contains an open neighborhood V of y that is path connected in U and hence in J. Since the open connected set V lies in J, V is the interior of an arc in J containing y. So $A = J - V$ is an arc in J containing x. Denote the endpoints of A by a and b. Thus, there exists a 1-1 map $p: I \to A$, where $p(0) = a$ and $p(1) = b$. Since $x \in A$ and $a \neq x \neq b$, there exists $x' \in (0,1)$ such that $p(x') = x$. Let A' be the arc joining a and x, and A'' the arc joining x and b so that $A = A' \cup A''$. Now $A' \cap Fr(U(x,\epsilon))$ is compact and does not contain x.
Let $L = \{t \in I \mid t < x' \text{ and } p(t) \in A' \cap Fr(U(x,\epsilon))\}$. Since $Fr(U(x,\epsilon))$ is a Jordan curve and A' contains points in both complementary domains of $Fr(U(x,\epsilon))$, L is nonempty. Since p is continuous, L is closed. Furthermore, since L is closed and bounded, it is

compact. Thus, L contains its least upper bound, say r. Similarly, $M = \{t \in I | x' < t \text{ and } p(t) \in A'' \cap \mathrm{Fr}\,(U(x,\epsilon))\}$ contains its greatest lower bound, say s. For all $t \in (r,s)$, $p(t) \in A \cap U(x,\epsilon)$ and $p(r), p(s) \in \mathrm{Fr}(U(x,\epsilon))$. Now $\mathrm{Fr}(U(x,\epsilon)) \cong S^1$ while $K = p[r,s] \cong [r,s] \cong [-1,1]$ since closed intervals are homeomorphic. Since K intersects $\mathrm{Fr}(U(x,\epsilon))$ at exactly two distinct points, $p(r)$ and $p(s)$, $K \cup \mathrm{Fr}(U(x,\epsilon))$ is a theta curve. □

<u>THEOREM 1.7</u>: Let $J \subset E^2$ be a Jordan curve and A one of its complementary domains. Given $x \in J$ and $\epsilon > 0$, there exists $\delta > 0$ such that $p,q \in A \cap U(x,\delta)$ can be joined by a path in $A \cap U(x,\epsilon)$.

PROOF: We may assume $\epsilon <$ diam J. By Lemma 1.6, there exists an arc K in J such that $x \in J$ and $K \cup \mathrm{Fr}(U(x,\epsilon))$ is a theta curve. Choose $\delta = \min\{\frac{\epsilon}{2}, d(x,J-\mathrm{Int}(K))\}$ and let $p,q \in U(x,\delta) \cap A$. Let $B = (J-\mathrm{Int}(K) \cup \mathrm{Fr}(U(x,\epsilon))$. Now B is connected, and since $U(x,\delta)$ is a convex subset of $E^2 - B$, B fails to separate p from q. Also since $p,q \in A$, J fails to separate p from q. Now $B \cap J = J - \mathrm{Int}(K)$ is connected. Therefore by Theorem 1.4, $B \cup J = J \cup \mathrm{Fr}(U(x,\epsilon))$ fails to separate p from q. Therefore, p and q may be joined by a path in $E^2 - (J \cup \mathrm{Fr}(U(x,\epsilon)))$ and hence in $A \cap U(x,\epsilon)$. □

<u>THEOREM 1.8:</u> (Accessibility) Let $J \subset E^2$ be a Jordan curve and A one of its complementary domains. If $x \in J$, there exists a map $p : I \to A \cup \{x\}$ with $p[0,1) \subset A$ and $p(1) = x$.

PROOF: By Theorem 1.7, if i is a positive integer, there exists $\delta_i > 0$ such that if $p, q \in A \cap U(x,\delta_i)$, then p and q may be joined by a path in $A \cap U(x, \frac{1}{i})$. Inductively, we may assume $\delta_1 > \delta_2 > \cdots$. Let $x_1 \in A \cap U(x,\delta_1)$ and for $i = 1, 2, \cdots$, let $x_{i+1} \in A \cap U(x,\delta_i)$; let $p_i:[1 - \frac{1}{2^{i-1}}, 1 - \frac{1}{2^i}] \to A \cap U(x, \frac{1}{i})$ be a map such that $p_i(1 - \frac{1}{2^{i-1}}) = x_i$ and $p_i(1 - \frac{1}{2^i}) = x_{i+1}$. Define $p: I \to A \cup \{x\}$ by

$$p(t) = \begin{cases} p_i(t) & \text{if } 1 - \frac{1}{2^{i-1}} \le t \le 1 - \frac{1}{2^i} \\ x & \text{if } t = 1 \end{cases} \quad . \ \square$$

The final result of this chapter will be vital to the result of Chapter III. A definition and two lemmas precede its proof.

<u>DEFINITION 1.9</u>: A polygon J in E^2 is a <u>convex</u> polygon if J is the frontier of a convex domain.

<u>LEMMA 1.10</u>: Let J and L be convex polygons in E^2. If h is a homeomorphism of J onto L, then there exists a homeomorphism h^* of $J \cup J_B$ onto $L \cup L_B$ such that $h^*|J = h$.

The lemma follows by selecting $a \in J_B$, $b \in L_B$ and mapping the line segment from a to $x \in J$ linearly onto the line segment joining b and $h(x)$. Details shall be omitted here. $\square$

<u>LEMMA 1.11</u>: Let J be a polygon in E^2 and L a triangle in E^2. If h is a homeomorphism of J onto L, then there exists a homeomorphism h^* of $J \cup J_B$ onto $L \cup L_B$ such that $h^*|J = h$.

PROOF: We will give an inductive proof on the number of line segments forming J.

If $J_1 \subset E^2$ is a triangle, then J_1 is convex, and the lemma follows from Lemma 1.10.

Now suppose that if $J_k \subset E^2$ is a polygon having k or fewer sides and h_k is a homeomorphism of J_k onto L, then there exists a homeomorphism $(h_k)^*$ of $J_k \cup (J_k)_B$ onto $L \cup L_B$ such that $(h_k)^*|J = h_k$.

Let $J_{k+1} \subset E^2$ be a polygon having $k + 1$ sides, and let h_{k+1} be a homeomorphism of J_{k+1} onto L. We may assume J_{k+1} is not convex, for otherwise the desired conclusion follows from Lemma 1.10. On page 286 of [4], Einar Hille shows that every polygon can be triangulated. In his proof, he shows that from the vertex p of an interior angle of J_{k+1}, having measure greater than π, there exists a line segment Q to some other vertex r of J_{k+1} such that $\text{Int}(Q)$ lies completely in $(J_{k+1})_B$, $Q \cap J_{k+1} = \{p,r\}$, and Q is a diagonal subdividing J_{k+1} into two polygons F and G, each having at most k sides. (Note that $F \cap G = Q$.) Let L' be a triangle in E^2. Let p' be a vertex of L', r' the midpoint of the opposite side, and Q' the median of L' having p' and r' as its endpoints. Now Q' lies completely in L'_B, $Q' \cap L' = \{p',r'\}$, and Q' is a diagonal subdividing L' into two triangles F' and G'. (Note that $F' \cap G' = Q'$.) Let m be a homeomorphism of J_{k+1} onto L' such that $m(p) = p'$ and $m(r) = r'$. Now there exist homeomorphisms f of F onto F' and g of G onto G' such that

$f|Cl(J_{k+1} - G) = m|Cl(J_{k+1} - G)$, $g|Cl(J_{k+1} - F) = m|Cl(J_{k+1} - F)$, and $f|Q = g|Q$. By our inductive hypothesis, there exist homeomorphisms f^* of $F \cup F_B$ onto $F' \cup F'_B$ and g^* of $G \cup G_B$ onto $G' \cup G'_B$ such that $f^*|F = f$ and $g^*|G = g$. By the gluing lemma, we have a homeomorphism m^* of $J_{k+1} \cup (J_{k+1})_B$ onto $L' \cup L'_B$ defined by

$$m^*(x) = \begin{cases} f^*(x) & \text{if } x \in F \cup F_B \\ g^*(x) & \text{if } x \in G \cup G_B \end{cases}$$

Now $m^*|Cl(J_{k+1} - G) = f^*|Cl(J_{k+1} - G) = f|Cl(J_{k+1} - G) = m|Cl(J_{k+1} - G)$ and $m^*|Cl(J_{k+1} - F) = g^*|Cl(J_{k+1} - F) = g|Cl(J_{k+1} - F) = m|Cl(J_{k+1} - F)$. Thus, since $Cl(J_{k+1} - F) \cup Cl(J_{k+1} - G) = J_{k+1}$, $m^*|J_{k+1} = m$.
Now $h_{k+1} \circ m^{-1}$ is a homeomorphism of L' onto L, and by Lemma 1.10 there exists a homeomorphism $(h_{k+1} \circ m^{-1})^*$ of $L' \cup L'_B$ onto $L \cup L_B$ such that $(h_{k+1} \circ m^{-1})^*|L' = h_{k+1} \circ m^{-1}$. Let the homeomorphism $(h_{k+1})^*$ of $J_{k+1} \cup (J_{k+1})_B$ onto $L \cup L_B$ be defined by $(h_{k+1})^* = (h_{k+1} \circ m^{-1})^* (m)^*$ so that $(h_{k+1})^*|J_{k+1} = h_{k+1}$.

The lemma now follows by induction. □

<u>THEOREM 1.12</u>: Let J, K be two polygons in E^2. If h is a homeomorphism of J onto K, then there exists a homeomorphism h^* of $J \cup J_B$ onto $K \cup K_B$ such that $h^*|J = h$.

PROOF: Let L be a triangle in E^2, and let h' be a homeomorphism of J onto L. By Lemma 1.11, there exists a homeomorphism $(h')^*$ of $J \cup J_B$ onto $L \cup L_B$ such that $(h')^*|J = h'$. Now $h' \circ h^{-1}$ is a homeomorphism of K onto L, and by Lemma 1.11 there exists a homeomorphism $(h'\circ h^{-1})^*$ of $K \cup K_B$ onto $L \cup L_B$ such that $(h'\circ h^{-1})^*|K = h'\circ h^{-1}$. Let the homeomorphism h^* of $J \cup J_B$ onto $K \cup K_B$ be defined by $h^* = [(h'\circ h^{-1})]^{-1}(h')^*$ so that $h^*|J = h$. $\square$

CHAPTER II

A PROOF OF THE GENERALIZED SCHOENFLIES THEOREM

The following problem was of interest for some time: Suppose h is a homeomorphic embedding of $S^{n-1} \times I$ in S^n. Are the closures of the complementary domains of $h(S^{n-1} \times \{\frac{1}{2}\})$ topological n-cells? With an added "niceness" condition on h, Mazur [6] gave an affirmative answer. With Mazur's result, Morse [7] proved the theorem without the "nice" condition. Morton Brown [2] proved directly that the answer is affirmative with no extra conditions on h required.

In this chapter, we shall present Brown's proof of the generalized Schoenflies Theorem. Theorems 2.3, 2.4, 2.8, 2.10, 2.11, and 2.12 are taken directly from [2], and the proofs are just embellished versions of Brown's proofs.

<u>DEFINITION 2.1</u>: If $f:X \to Y$ is a map, then an <u>inverse</u> <u>set</u> (under f) is a set $M \subset X$ containing at least two points, and such that for some y of $f(X)$, $M = f^{-1}(y)$.

<u>DEFINITION 2.2</u>: A set M is <u>cellular</u> in an n-dimensional metric space S if there exists n-cells $Q_1, Q_2, \ldots$ in S such that $Q_{i+1} \subset \mathrm{Int}(Q_i)$, and $\bigcap_{i=1}^{\infty} Q_i = M$.

<u>THEOREM 2.3</u>: Let Q be an n-cell, and let f map Q into the n-sphere S^n. Suppose also that f has only a finite number of inverse

sets, and that these inverse sets are all in $\mathrm{Int}(Q)$. Then $f(Q)$ is the union of $f(\mathrm{Fr}(Q))$ and one of its complementary domains.

PROOF: Let D and E be the complementary domains of $f(\mathrm{Fr}(Q))$ in S^n. Let $h = f|\mathrm{Fr}(Q)$. If $f(Q) \subset f(\mathrm{Fr}(Q))$, then $h^{-1}f$ maps Q into $\mathrm{Fr}(Q)$ and is fixed on $\mathrm{Fr}(Q)$. This is impossible; hence we may assume $f(Q) \cap D \neq \phi$. Now $\mathrm{Fr}(Q)$ does not separate Q, and $f(\mathrm{Fr}(Q))$ separates S^n. Since $\mathrm{Int}(Q)$ is path connected, $f(\mathrm{Int}(Q))$ is path connected. Since $f(\mathrm{Fr}(Q)) \cap D = \phi$ and $f(Q) \cap D \neq \phi$, $f(\mathrm{Int}(Q)) \cap D \neq \phi$. Suppose $f(\mathrm{Int}(Q)) \not\subset \mathrm{Cl}(D)$. Then there exists an element $p \in f(\mathrm{Int}(Q))$ such that $p \notin \mathrm{Cl}(D)$; i.e., $p \in E$. Let $q \in f(\mathrm{Fr}(Q)) \cap D$. There exists a path m joining p and q in $f(\mathrm{Int}(Q))$. This path intersects $f(\mathrm{Fr}(Q))$. Let $x \in f(\mathrm{Fr}(Q)) \cap f(\mathrm{Int}(Q)) \neq \phi$. Then $f^{-1}(x)$ intersects $\mathrm{Fr}(Q)$ and $\mathrm{Int}(Q)$, but the only inverse sets of f are in $\mathrm{Int}(Q)$. Therefore, $f(\mathrm{Int}(Q)) \subset \mathrm{Cl}(D)$, and so $f(Q) \subset \mathrm{Cl}(D)$.

Let $A_1, A_2, \ldots, A_j$ be the inverse sets under f. These all lie in $\mathrm{Int}(Q)$. By Brouwer's Invariance of Domain Theorem, f maps the interior of $Q - \bigcup_{i=1}^{j} A_i$ homeomorphically onto an open connected set U in S^n. Let $V = D - (U \cup f(\bigcup_{i=1}^{j} A_i))$. Since $V = D - f(Q)$ and $f(Q)$ is compact, V is an open subset of D, as is U. Now, $D = U \cup V \cup f(\bigcup_{i=1}^{j} A_i)$. If U and V are both nonempty, then this implies that $f(\bigcup_{i=1}^{j} A_i)$, a finite set, separates D. This is a contradiction. Hence $V = \phi$ which implies $D \subset U \cup f(\bigcup_{i=1}^{j} A_i)$. Thus $\mathrm{Cl}(D) \subset \mathrm{Cl}(U \cup f(\bigcup_{i=1}^{k} A_i))$ $\subset \mathrm{Cl}(f(Q))$.

It follows that $f(Q) = \mathrm{Cl}(D)$. □

<u>THEOREM 2.4</u>: Let Q be an n-cell. Suppose M is a cellular subset of $Int(Q)$. Then there is a map f of Q onto itself such that f is fixed on $Fr(Q)$ and M is the only inverse set under f.

We shall precede the proof of Theorem 2.4 by some lemmas required for that proof.

<u>LEMMA 2.5</u>: Let Q be an n-cell, and let A be a compact set lying in $Int(Q)$. Then if $\epsilon > 0$, there exists a homeomorphism $h:Q \to Q$ such that

1) $h(x) = x$ for all $x \in Fr(Q)$, and

2) $\operatorname{diam} h(A) < \epsilon$.

PROOF: Since closed intervals are homeomorphic, we may assume $Q = I^n$. Let $0 < \delta < \epsilon$ and $t = d(A,Fr(Q)) = \text{l.u.b.}\{|a-b| \,|\, a \in A, b \in Fr(Q)\}$. Define $g_1:[0, 1-t] \to [0,\frac{\delta}{\sqrt{n}}]$ by $g_1(x) = \frac{\delta}{\sqrt{n}(1-t)}x$ for all $x \in [0, 1-t]$, and define $g_2:[1-t, 1] \to [\frac{\delta}{\sqrt{n}}, 1]$ by $g_2(x) = \frac{n-\sqrt{n}\delta}{nt}(x-1) + 1$ for all $x \in [1-t, 1]$ so that the map $f:I \to I$ defined by

$$f(x) = \begin{cases} g_1(x) & \text{if } x \in [0, 1-t] \\ g_2(x) & \text{if } x \in [1-t, 1] \end{cases}$$

is a homeomorphism. Let $\overline{x} = (x_1, x_2, \ldots, x_n)$ be an n-tuple in Q, and define $h:Q \to Q$ by $h(\overline{x}) = h(x_1, x_2, \ldots, x_n) = (f(x_1), f(x_2), \ldots, f(x_n))$. Let $\overline{y} = (y_1, y_2, \ldots, y_n)$ be an n-tuple in Q such that $\overline{x} \neq \overline{y}$, and suppose $h(\overline{x}) = h(\overline{y})$. Then $(f(x_1), f(x_2), \ldots, f(x_n)) = (f(y_1), f(y_2), \ldots, f(y_n))$. So $f(x_i) = f(y_i)$ for all $i = 1, 2, \ldots, n$, and since f is a homeomorphism, $x_i = y_i$ for all $i = 1, 2, \ldots, n$. This is a contradiction. Thus $h(\bar{x}) \neq h(\bar{y})$, and h is 1-1. It can also be seen that h is onto since for $\bar{x} \in Q$, $h(f^{-1}(x_1), f^{-1}(x_2), \ldots, f^{-1}(x_n)) = \bar{x}$. Therefore h is a homeomorphism, and $h(x) = x$ if $x \in Fr(Q)$. Since $h(A) \subset [0, \frac{\delta}{\sqrt{n}}]^n$, diam $h(A) \leq \delta < \epsilon$. □

<u>LEMMA 2.6</u>: Let Q be an n-cell. Suppose M is a cellular subset of $Int(Q)$. Let Q_i be a sequence of n-cells in Q whose intersection is M and such that $Q_{i+1} \subset Int(Q_i)$. Let $id = h_0, h_1, h_2, \ldots, h_n$ be homeomorphisms of Q onto itself such that

1) $h_{i+1} = h_i$ on $Q-Q_i$ for $i = 1, 2, \ldots, n-1$, and
2) diam $h_i(Q_i) < \frac{1}{i}$ for $i = 1, 2, \ldots, n$.

Then $\{h_i\}$ converges uniformly on Q.

PROOF: Let $\epsilon > 0$. By the Archimedean principle, there exists a positive integer N such that $\frac{1}{N} < \epsilon$. By our hypothesis, for all $x \in Q$ and $j,k \geq N$, $|h_j(x) - h_k(x)| < \frac{1}{N} < \epsilon$. Therefore, by the Cauchy criterion for uniform convergence, $\{h_i\}$ converges uniformly on Q. □

LEMMA 2.7: With notation as above, let $\lim_{i\to\infty} h_i = f$. Then the following conditions hold:

1) f is continuous,
2) $f(M) = \cap\, h_i(Q_i)$ is a point,
3) $f|Q-M$ is 1-1, and
4) $f(Q) = Q$.

PROOF: 1) Let $\epsilon > 0$. Since $\{h_i\}$ converges uniformly on Q to f, there exists a positive integer N such that $|h_N(x) - f(x)| < \frac{\epsilon}{3}$ for all $x \in Q$. Let $a \in Q$. Then

$$|f(x) - f(a)| = |f(x) - h_N(x) + h_N(x) - h_N(a) + h_N(a) - f(a)|$$

$$\leq |f(x) - h_N(x)| + |h_N(x) - h_N(a)| + |h_N(a) - f(a)|$$

$$< \frac{\epsilon}{3} + |h_N(x) - h_N(a)| + \frac{\epsilon}{3} \quad \text{for all } x \in Q.$$

Since h_N is continuous at a, there exists $\delta > 0$ such that if $|x-a| < \delta$, then $|h_N(x) - h_N(a)| < \frac{\epsilon}{3}$. So we have $|f(x) - f(a)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$ if $|x-a| < \delta$. Therefore f is continuous.

2) Since $h_1(Q_1) \supset h_2(Q_2) \supset \ldots \supset h_i(Q_i) \supset \ldots$ is a sequence of nonempty subsets of Q, it follows from the Cantor Intersection Theorem [1] that there exists a point belonging to $\cap\, h_i(Q_i)$. Furthermore, since $\{\text{diam}\,(h_i(Q_i))\} \to 0$, $\cap\, h_i(Q_i)$ contains precisely one point. Suppose $y \in f(M)$. Then there exists $x \in M$ such that $f(x) = y$. Since $M \subset Q_i$ for all i, $x \in Q_i$ for all i. Since $\lim_{i\to\infty} h_i(x) = f(x) = y$, $\{h_i(x)\} \to y$. Suppose now that $y \notin h_i(Q_i)$. Then there exists a positive integer j for which $y \notin h_j(Q_j)$. So $y \notin h_k(Q_k)$ for all $k \geq j$. Choose $\epsilon = d(y, h_j(Q_j))$, and let $U(y,\epsilon)$ be the ball about y of radius ϵ. Now $U(y,\epsilon)$ contains at most a finite number of terms of $\{h_i(x)\}$, but this is a contradiction. Thus $f(M) \subset \cap\, h_i(Q_i)$. Now since $\cap\, h_i(Q_i)$ contains only one point and $f(M) \neq \phi$, $f(M) = \cap\, h_i(Q_i)$.

3) Suppose $a,b \in Q-M$ and $a \neq b$. Then there exist positive integers j and k such that $a \in Q - Q_j$ and $b \in Q - Q_k$. Choose

$i = \max\{j,k\}$ so that $a,b \in Q - Q_i$. Since h_i is a homeomorphism, $h_i(a) \neq h_i(b)$. Since $h_i|Q - Q_i = f|Q - Q_i$, $f(a) \neq f(b)$. Therefore $f|Q - M$ is $1 - 1$.

4) Since f is a map of Q onto itself, $f(Q) \subset Q$. Suppose $x \in Q = \cap h_i(Q_i) \cup (Q - \cap h_i(Q_i))$. If $x \in \cap h_i(Q_i)$, then $x \in f(Q)$ since $\cap h_i(Q_i) = f(M) \subset f(Q)$. Let $x \in Q - \cap h_i(Q_i)$. Then there exists a positive integer j such that $x \notin h_j(Q_j)$. Let k be the least such positive integer. Now there exists $y \in Q$ such that $h_k(y) = x$. For each $n \geq k$, $h_n(y) = h_{n-1}(y) = \ldots = h_{k+1}(y) = h_k(y)$. So $x = h_k(y) = \lim_{n\to\infty} h_n(y) = f(y) \in f(Q)$. Therefore $f(Q) = Q$. □

<u>PROOF OF THEOREM 2.4</u>: Let Q_i be a sequence of n-cells in Q whose intersection is M and such that $Q_{i+1} \subset \mathrm{Int}(Q_i)$. From Lemma 2.5, there exists a homeomorphism h_1 of Q onto itself which is fixed on $\mathrm{Fr}(Q)$ and such that the diameter of $h_1(Q_1)$ is less than 1. Inductively, suppose we have homeomorphisms $id = h_0, h_1, \ldots, h_n$ of Q onto itself such that

1) $h_{i+1} = h_i$ on $Q - Q_i$ for $i = 1, 2, \ldots, n-1$, and
2) $\mathrm{diam}\,(h_i(Q_i)) < \frac{1}{i}$ for $i = 1, 2, \ldots, n$.

By Lemma 2.7 (1 and 4), $f = \lim_{i\to\infty} h_i$ is a map of Q onto itself. The h_i are all fixed on $\mathrm{Fr}(Q)$; hence so is f.

Let x,y be two different points of Q, one in M and one in $Q - M$. We may assume $x \notin M$. So for some i, $x \notin Q_i$ and $f(x) = h_i(x) \notin h_i(Q_i)$. Now since $y \in M$, then $f(y) \in h_i(\mathrm{Int}(Q_i)) \subset h_i(Q_i)$. So $f(x) \neq f(y)$.

Let x,y be two different points of $Q - M$. By Lemma 2.7 (3), $f(x) \neq f(y)$.

Let $x,y \in M$. By Lemma 2.7 (2), $f(M)$ is a point, so $f(x) = f(y)$. Therefore, M is the only inverse set under f. □

<u>THEOREM 2.8</u>: Let S be a topological n-1 sphere in S^n, and let D be one of its complementary domains. Suppose f maps $Cl(D)$ onto an n-cell R such that the only inverse set of f is a cellular subset M of D. Then $Cl(D)$ is an n-cell.

PROOF: Let Q be an n-cell in D such that $M \subset Int(Q)$. Then M is cellular in Q. By Theorem 2.4, there exists a map g of Q onto itself such that g is fixed on $Fr(Q)$, and M is the only inverse set under g. Let g' be the map of $Cl(D)$ onto itself which is the identity on $Cl(D) - Q$ and g on Q. Then the map fg'^{-1} of $Cl(D)$ onto R is a homeomorphism. Therefore $Cl(D)$ is an n-cell. □

<u>LEMMA 2.9</u>: Let Q be an n-cell, and suppose f maps Q into S^n. Suppose that $M \subset Int(Q)$ is the only inverse set under f. There is a complementary domain D of $f(Fr(Q))$ such that if U is an open subset of $Int(Q)$ containing M, then $f(U)$ is open in D.

PROOF: By Theorem 2.3, there is a complementary domain D of $f(Fr(Q))$ such that $f(Q) = f(Fr(Q)) \cup D$. Since $U - M$ is open, $f(U-M)$ is open by Brouwer's Invariance of Domain Theorem. Suppose $f(M) = \{m\}$ is not an interior point of $f(U)$. For every positive integer i, there exists $y_i \in U(m,\frac{1}{i}) - f(U)$. We may assume $1 < d(m,f(Fr(Q)))$.

Now the sequence $\{f^{-1}(y_n)\}$ lies completely in $Q - U$ and therefore has a subsequential limit point, say p, in $Q - U$. So there exists a subsequence $\{f^{-1}(y_{n_i})\}$ of $\{f^{-1}(y_n)\}$ which converges to p. Since f is continuous at p, then

$$f(p) = \lim_{f^{-1}(y_{n_i})\to p} f(f^{-1}(y_{n_i})) = m \quad ;$$

but this is a contradiction, since $p \in U$. Therefore $f(U)$ is open in D. $\square$

THEOREM 2.10: Let Q be an n-cell, and suppose f maps Q into S^n. Suppose also that $M \subset \mathrm{Int}(Q)$ is the only inverse set under f. Then M is cellular in Q.

PROOF: By Theorem 2.3, there is a complementary domain D of $f(\mathrm{Fr}(Q))$ so that $f(Q) = f(\mathrm{Fr}(Q)) \cup D$. Let U be an open subset of $\mathrm{Int}(Q)$ containing M. Then by Lemma 2.9, $f(U)$ is open in D, and $f(M) \subset f(U)$. Let h be a homeomorphism of S^n into itself such that $h(\mathrm{Cl}(D)) \subset f(U)$, and for some small neighborhood V of the point of $f(M)$, $h|V$ is fixed. Let the map g of Q onto itself be defined by

$$g(x) = \begin{cases} x & \text{if } x \in m \\ f^{-1}hf(x) & \text{if } x \in M \end{cases} .$$

Since $f^{-1}hf$ is the identity on $f^{-1}(V)$, g is a well defined homeomorphism. Therefore $g(Q)$ is an n-cell in U containing M in its interior. It follows that M is cellular in Q. $\square$

THEOREM 2.11: Let f map S^n onto itself such that f has precisely two inverse sets A and B. Then both A and B are cellular in S^n.

PROOF: Let d be an $n-1$ sphere in $S^n - (A \cup B)$ each of whose complementary domains has an n-cell for its closure. If d separates A from B, then the theorem follows from Theorem 2.10. Suppose d does not separate A from B. Let Q be the n-cell whose frontier is d with $A \cup B \subset \mathrm{Int}(Q)$. Let $f(A) = \{a\}$ and $f(B) = \{b\}$. By Theorem 2.3, $f(Q) = f(\mathrm{Fr}(Q)) \cup D$ where D is that complementary domain of $f(\mathrm{Fr}(Q))$ such that $(\{a\} \cup \{b\}) \subset D$. Let U be an open subset of D which contains $\{a\}$ but not $\{b\}$. Let h be a homeomorphism of S^n onto itself such that $h(f(Q)) \subset U$, and for some small neighborhood V of a, $h|V$ is fixed. Let g be the map of Q into itself defined by

$$g(x) = \begin{cases} x & \text{if } x \in A \\ f^{-1}hf(x) & \text{if } x \in Q - A \end{cases} .$$

Since $f^{-1}hf$ is the identity on $f^{-1}(V)$, g is a well defined map. Now the only inverse set of g is B. So by Theorem 2.10, B is cellular in S^n. Similarly, A is cellular in S^n. □

THEOREM 2.12: (THE GENERALIZED SCHOENFLIES THEOREM) Let h be a homeomorphic embedding of $S^{n-1} \times I$ into S^n. Then the closure of either complementary domain of $h(S^{n-1} \times \{\frac{1}{2}\})$ is an n-cell.

PROOF: Let A be the closure of the complementary domain of $h(S^{n-1} \times \{1\})$ which does not contain $h(S^{n-1} \times \{0\})$. Let B be the closure of the complementary domain of $h(S^{n-1} \times \{0\})$ which does not contain $h(S^{n-1} \times \{1\})$. Let f be a map of S^n onto itself which carries A onto the north pole, B onto the south pole, and $h(S^{n-1} \times \{\frac{1}{2}\})$ onto the equator, and has only A and B as inverse sets. Let D_A and D_B be the complementary domains of $h(S^{n-1} \times \{\frac{1}{2}\})$ which contain A and B, respectively. By Theorem 2.11, A and B are cellular in S^n. Thus, A and B are cellular in D_A and D_B, respectively. Therefore by Theorem 2.8, $Cl(D_A)$ and $Cl(D_B)$ are n-cells. $\square$

CHAPTER III

THE CLASSICAL SCHOENFLIES THEOREM

Our claim is that every Jordan curve in E^2 satisfies the collaring hypothesis of Brown's generalized Schoenflies Theorem. It will suffice to show that given any Jordan curve J in E^2, there exists an embedding $h':S^1 \times I \to E^2$ such that $h'(S^1 \times \{\frac{1}{2}\}) = J$. We shall show there exists an embedding $h:S^1 \times I \to J \cup J_B$ such that $h(S^1 \times \{1\}) = J$ and then with that result expand to an h'. Our embedding h will be defined as the limit of a sequence of functions converging uniformly on $S^1 \times I$ which will be obtained inductively. First, however, a number of preliminary definitions, lemmas, and theorems are given.

DEFINITION 3.1: A polygonal path in the piecewise linear space X is a path p such that $p(I)$ is a singleton or the union of a finite number of line segments.

DEFINITION 3.2: A polygonal arc q in the piecewise linear space X is the union of a finite number of line segments such that q is homeomorphic to I.

DEFINITION 3.3: A set A in the piecewise linear space X is polygonally path connected if for any two points $a,b \in A$, a and b may be joined by a polygonal path whose image lies in A.

<u>DEFINITION 3.4</u>: A set A in the piecewise linear space X is <u>polygonally arc connected</u> if for any two points $a,b \in A$, a and b may be joined by a polygonal arc lying in A.

<u>THEOREM 3.5</u>: A nonempty connected set A in E^2 is polygonally path connected.

PROOF: Let $x \in A$ and let $B = \{a \in A | x$ and a can be joined by a polygonal path in $A\}$. (Note that $B \neq \phi$ since $x \in B$.) Let $y \in B$. Since $y \in A$ and A is open, there exist $\epsilon > 0$ such that $U(y,\epsilon) \subset A$. Now let $z \in U(y,\epsilon)$, and let the map $g:[\frac{1}{2}, 1] \to U(y,\epsilon)$ be defined by $g(t) = 2[(1-t)y + (t - \frac{1}{2})z]$ for all $t \in [\frac{1}{2},1]$. Since $y \in B$, there exists a polygonal path $h:[0,\frac{1}{2}] \to A$ such that $h(0) = x$ and $h(\frac{1}{2}) = y$. Now define the map $p:I \to A$ by

$$p(t) = \begin{cases} h(t), & 0 \leq t \leq \frac{1}{2} \\ g(t), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

so that p is a polygonal path in A joining x and z, implying $z \in B$. So $U(y,\epsilon) \subset B$, and thus B is open relative to A.

Now let y be a limit point of B in A. Since $y \in A$ and A is open, there exist $\epsilon > 0$ such that $U(y,\epsilon) \subset A$. Since y is a limit point of B, there exists a point $w \in B \cap U(y,\epsilon)$. Since $w \in B$, there exists a polygonal path $m:[0,\frac{1}{2}] \to A$ such that $m(0) = x$ and $m(\frac{1}{2}) = w$. Let the map $n:[\frac{1}{2},1] \to U(y,\epsilon)$ be defined by $n(t) = 2[(1-t)w + (t-\frac{1}{2})y]$ for all $t \in [\frac{1}{2},1]$. Now define the map $q:I \to A$ by

$$q(t) = \begin{cases} m(t), & 0 \le t \le \frac{1}{2} \\ n(t), & \frac{1}{2} \le t \le 1 \end{cases} .$$

Then q is a polygonal path in A joining x and y. So $y \in B$, and B is closed relative to A. Therefore $A = B$.

Let $a,b \in A$. By the above, there exists a polygonal path $r_1:[0,\frac{1}{2}] \to A$ such that $r_1(0) = a$ and $r_1(\frac{1}{2}) = x$; also, there exists a polygonal path $r_2:[\frac{1}{2},1] \to A$ such that $r_2(\frac{1}{2}) = x$ and $r_2(1) = b$. Define $r:I \to A$ by

$$r(t) = \begin{cases} r_1(t), & 0 \le t \le \frac{1}{2} \\ r_2(t), & \frac{1}{2} \le t \le 1 \end{cases} .$$

Then r is a polygonal path in A joining a and b. It follows that A is polygonally path connected. □

<u>REMARK 3.6</u>: Obviously the above result holds if E^2 is replaced by any piecewise linear space X having the following property: Each point of X has arbitrarily small open polygonally path connected neighborhoods.

<u>THEOREM 3.7, [1] (LEBESGUE Covering Theorem)</u>: Suppose $\mathcal{G} = \{G_\alpha\}$ is an open covering of a compact subset K of E^n. There exists a positive number λ such that if x,y belong to K and $|x-y| < \lambda$, then there is a set in $\mathcal{G}$ containing both x and y. □

<u>LEMMA 3.8</u>: Let $a,b \in E^2$ and $\epsilon > 0$. If $p:I \to E^2$ is a path in E^2 such that $p(0) = a$ and $p(1) = b$, then there exists a path

$q: I \to E^2$ with $q(0) = a$ and $q(1) = b$ such that

1) q is polygonal and

2) for all $y \in I$, $|p(y)-q(y)| < \epsilon$.

PROOF: For $x \in p(I)$, let $E_x = U(x, \frac{\epsilon}{4})$ and $F = \{E_x | x \in p(I)\}$. By Theorem 3.7, there exists $\lambda > 0$ such that if $c, d \in p(I)$ and $|c-d| < \lambda$, then there exists a set in F containing both c and d. Since p is uniformly continuous, there exists $\delta > 0$ such that if $e, f \in I$ and $|e-f| < \delta$, then $|p(e)-p(f)| < \lambda$. Let $x_1, x_2, \ldots, x_n$ be a partition of I such that $0 = x_1 < x_2 < \ldots < x_n = 1$ and $|x_i - x_{i+1}| < \delta$ for $i = 1, 2, \ldots, n-1$. Define the map $q_i : [x_i, x_{i+1}] \to E^2$, for $i = 1, 2, \ldots, n-1$, by $q_i(t) = \frac{1}{x_{i+1}-x_i} \cdot \{[x_{i+1}-t][p(x_i)] + [t-x_i][p(x_{i+1})]\}$ for all $t \in [x_i, x_{i+1}]$. Now define the map $q: I \to E^2$ by $q(t) = q_i(t)$ if $t \in [x_i, x_{i+1}]$. Let $y \in I$. There exists a positive integer k, with $1 \le k \le n-1$, such that $y \in [x_k, x_{k+1}]$. Now since $|x_k - y| < \delta$, $|p(x_k) - p(y)| < \lambda$. So there exists a set A in F containing both $p(x_k)$ and $p(y)$, and therefore $|p(y) - p(x_k)| < \frac{\epsilon}{2}$. Since $|x_k - x_{k+1}| < \delta$, $|p(x_k) - p(x_{k+1})| < \lambda$. So there exists a set A' in F containing both $p(x_k)$ and $p(x_{k+1})$, and therefore $|p(x_k) - p(x_{k+1})| < \frac{\epsilon}{2}$. Furthermore $|p(x_k) - q(y)| \le |p(x_k) - p(x_{k+1})|$ since $q(y)$ lies on the line segment joining $p(x_k)$ and $p(x_{k+1})$. Now we have

$$\begin{aligned} |p(y) - q(y)| &= |p(y)-p(x_k)+p(x_k)-q(y)| \\ &\le |p(y)-p(x_k)|+|p(x_k)-q(y)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon . \end{aligned}$$

□

LEMMA 3.9: Let $J \subset E^2$ be a Jordan curve and let $x \in J$. Let A be one of the complementary domains of J. There exists a path $q:I \to A \cup \{x\}$ with $q[0,1) \subset A$ and $q(1) = x$ such that $q[0,1)$ is the union of countably many line segments such that for each $\epsilon > 0$, $U(x,\epsilon)$ contains all but a finite number of those line segments.

PROOF: By Theorem 1.8, there exists a map $p:I \to A \cup \{x\}$ such that $p[0,1) \subset A$ and $p(1) = x$. Let $\{x_n\}$ be an increasing sequence of points in $[0,1)$ converging to 1 such that $x_1 = 0$. Then $\{p(x_n)\} \to x$. Let $f_i:I \to [x_i,x_{i+1}]$ be the homeomorphism defined by $f_i(t) = (x_{i+1}-x_i)t + x_i$ for each $i \in Z^+$. Let $\epsilon_i = d(p[x_i,x_{i+1}],\ E^2 - A)$. From Lemma 3.8, for each $i \in Z^+$, there exists a map $q_i:I \to A$ such that $q_i(0) = (p\circ f_i)(0)$, $q_i(1) = (p\circ f_i)(1)$, q_i is polygonal, and for all $t \in I$, $|(p\circ f_i)(t) - q_i(t)| < \epsilon_i$. Define $q:I \to A \cup \{x\}$ by

$$q(t) = \begin{cases} (q_i\circ f_i^{-1})(t) & \text{if } x_i \le t \le x_{i+1} \\ x & \text{if } t = 1 \end{cases}$$

so that $q[0,1) \subset A$ and $q[0,1)$ is the union of countably many line segments such that for each $\epsilon > 0$, $U(x,\epsilon)$ contains all but a finite number of those line segments.

LEMMA 3.10: Let $J \subset E^2$ be a Jordan curve and A one of its complementary domains. Then given $x \in J$ and $\epsilon > 0$, there exists $\delta > 0$ such that two points $a,b \in A \cap U(x,\delta)$ can be joined by a polygonal path in $A \cap U(x,\epsilon)$.

PROOF: From Theorem 1.7, if $x \in J$ and $\epsilon > 0$, there exists $\delta > 0$ such that if $p,q \in A \cap U(x,\delta)$, there exists a path $p:I \to U(x,\epsilon)$ such that $p(0) = a$ and $p(1) = b$. Let $\gamma = d(p(I), Fr(A \cap U(x,\epsilon)))$. By Lemma 3.8, there exists a path $q:I \to E^2$ with $q(0) = a$ and $q(1) = b$ such that q is polygonal and for all $t \in I$, $|p(y)-q(y)|<\gamma$. It follows that q is a polygonal path joining a and b in $A \cap U(x,\epsilon)$. □

<u>LEMMA 3.11</u>: Let x and y be points of the piecewise linear space X and let $p:I \to X$ be a polygonal path such that $p(0) = x$ and $p(1) = y$. Then there exists a 1-1 polygonal path $q:I \to X$ such that $q(0) = x$, $q(1) = y$, and $q(I) \subset p(I)$.

PROOF: If $p_1:I \to X$ is a polygonal path such that $p_1(0) = x$, $p_1(1) = y$, and $p_1(I)$ is a line segment, then define $q_1:I \to X$ by $q_1(t) = (1-t)x + ty$ for all $t \in I$.

Suppose for any path $p_k:I \to X$ such that $p_k(0) = x$, $p_k(1) = y$, and $p_k(I)$ is the union of at most k line segments, there exists a 1-1 polygonal path $q_k:I \to X$ such that $q_k(0) = x$, $q_k(1) = y$, and $q_k(I) \subset p_k(I)$.

Let $p_{k+1}:I \to X$ be a polygonal path such that $p_{k+1}(0) = x$, $p_{k+1}(1) = y$, and $p_{k+1}(I)$ is the union of $k+1$ line segments, say $a_1, a_2, \ldots, a_{k+1}$. We may assume $p_{k+1}(0) \in a_1$. Since a_1 is closed, $p_{k+1}^{-1}(a_1)$ is closed and has a largest element, say s. Since $p_{k+1}[s,1] \subset \bigcup_{i=2}^{k+1} a_i$, there exists a 1-1 map $p_0\ [s,1] \to p_{k+1}[s,1]$ such that $p_0(s) = p_{k+1}(s)$ and $p_0(1) = y$ by our inductive hypothesis.

Let $z = \text{l.u.b.}\{p_0^{-1}(a_1 \cap p_0[s,1])\}$. If $z = 0$, let $q_{k+1} = p_0 : I \to X$. If $p_0(z) = x$, then $p_0|[z,1]$ is a 1-1 map such that $p_0(z) = x$, $p_0(1) = y$, and $p_0([z,1]) \subset p_{k+1}(I)$. By reparameterization, there exists a 1-1 polygonal path $q_{k+1} : I \to X$ such that $q_{k+1}(0) = x$, $q_{k+1}(1) = y$, and $q_{k+1}(I) \subset p_{k+1}(I)$. Consequently, assume $p_0(z) \neq x$. Let $q:[0,z] \to a_1$ be defined by $q(t) = \frac{1}{z}[(z-t)(x) + (t)(p_0(z))]$ for all $t \in [0,z]$. Now let the map $q_{k+1} : I \to p_{k+1}(I)$ be defined by

$$q_{k+1}(t) = \begin{cases} q(t) & \text{if } 0 \le t \le s \\ p(t) & \text{if } s \le t \le 1 \end{cases} .$$

Now q_{k+1} is a 1-1 polygonal path such that $q_{k+1}(0) = x$, $q_{k+1}(1) = y$, and $q_{k+1}(I) \subset p_{k+1}(I)$.

The lemma holds by induction. □

<u>COROLLARY 3.12</u>: A space in E^2 is polygonally path connected if and only if it is polygonally arc connected. □

<u>DEFINITION 3.13</u>: Let $a \in A$, where A is an arc in E^2 which is the closure of the union of countably many line segments. If for $\epsilon > 0$, $U(a,\epsilon)$ contains all but a finite number of those line segments, then A is <u>locally polygonal modulo a</u> .

<u>LEMMA 3.14</u>: Let $J \subset E^2$ be a Jordan curve and A one of its complementary domains. Let $x \in J$ and $w \in A$. There exists an arc $B \subset A \cup \{x\}$ having endpoints w and x such that $B \cap J = \{x\}$ and B is locally polygonal modulo x.

PROOF: By Lemma 3.9, there exists a path $q_1: I \to A \cup \{x\}$ with $q[0,1) \subset A$ and $q_1(1) = x$ such that $q_1(I)$ is the union of countably many line segments such that for each $\epsilon > 0$, $U(x,\epsilon)$ contains all but a finite number of those line segments. If $q_1(0) = w$, let $f = q_1$. If $q_1(0) \neq w$, let $r_1: I \to A$ be a polygonal 1-1 map such that $r_1(0) = w$ and $r_1(1) = q_1(0)$. Define $r_2: [0,\frac{1}{2}] \to I$ by $r_2(t) = 2t$ for all $t \in [0,\frac{1}{2}]$ and $q_2: [\frac{1}{2},1] \to I$ by $q_2(t) = 2t-1$ for all $t \in [\frac{1}{2},1]$. Now let $f: I \to A \cup \{x\}$ be defined by

$$f(t) = \begin{cases} (r_1 0 r_2)(t) & \text{if } 0 \le t \le \frac{1}{2} \\ (q_1 0 q_2)(t) & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

so that in any case f is a path in E^2 such that $f(0) = w$, $f[0,1) \subset A$, $f(1) = x$, and $f(I)$ is the union of countably many line segments such that for $\epsilon > 0$, $U(x,\epsilon)$ contains all but a finite number of those line segments.

Let $\{a_i\}$ be the sequence of points in I defined by $a_i = 1 - \frac{1}{i}$ for each $i \in Z^+$ so that $\{a_i\} \to 1$. For each $i \in Z^+$, $f\,[a_i, a_{i+1}]$ is a polygonal path in E^2 joining $f(a_i)$ and $f(a_{i+1})$, and by Lemma 3.11, there exists a 1-1 polygonal path $p_i: I \to f[a_i, a_{i+1}]$ such that $p_i(0) = f(a_i)$ and $p_i(1) = f(a_{i+1})$. By gluing, there exists a polygonal path $p': I \to E^2$ such that $p'(0) = w$, $p'(1) = x$, and $p'|[a_i, a_{i+1}]$ is 1-1. Order the arc $B_i = p'[a_i, a_{i+1}]$ from $p'(a_i)$ to $p'(a_{i+1})$. Let b_1 be the first point of B_1 lying in $\bigcup_{i=2}^{\infty} B_i$. Let j_1 be the largest integer such that $b_1 \in B_{j_1}$, and let b_2 be the first point of the arc B_{j_1} from b_1 to $p(a_{j_1+1})$ lying in $\bigcup_{i=j_1+1}^{\infty} B_i$. Continuing this process, we obtain arcs (or degenerate arcs) $B'_0, B'_1, B'_2, \ldots$, where B'_0 is the arc of B_1 joining w and b_1, such that for $i = 0,1,2,\ldots$, $B'_i \cap (\bigcup_{k=i+1}^{\infty} B'_k) = \{b_{i+1}\}$. Let $B = (\bigcup_{i=0}^{\infty} B'_i) \cup \{x\}$ so that $B \subset A \cup \{x\}$ is an arc having endpoints w and x, $B \cap J = \{x\}$, and B is locally polygonal modulo x. □

<u>LEMMA 3.15</u>: Let $J \subset E^2$ be a Jordan curve and A one of its complementary domains. Let $b \in J$, and let L be a polygon contained in A. (If $A = J_U$, restrict L so that $J \subset L_B$.) Then for some $a \in L$, there exists an arc C in $A \cup \{x\}$ joining a and b such that $C \cap L = \{a\}$, $C \cap J = \{b\}$, and C is locally polygonal modulo b.

PROOF: Let $x \in L$ and $b \in J$. Since $x \in A$, by Lemma 3.14, there exists an arc B in $A \cup \{x\}$ joining x and b such that $B-\{b\} \subset A$, $B \cap J = \{b\}$, and B is locally polygonal modulo b. Since B is an arc, there exists a 1-1 map $p:I \to B$ such that $p(0) = x$, $p(1) = b$, and $p(I) = B$. Since $B \cap L$ is closed, $p^{-1}(B \cap L)$ is closed and has a largest element, say s. Let $f:I \to [s,1]$ be the homeomorphism defined by $f(t) = (1-s)t + s$ for all $t \in I$. Define the map $q:I \to B$ by $q(t) = (p \circ f)(t)$ for all $t \in I$, and let $p(s) = a$ so that $q(I) = C$ is an arc satisfying the required conditions. □

<u>REMARK 3.16</u>: If $J \subset E^2$ is a Jordan curve, A is one of its complementary domains, $b_1 \in J$, and α_1 is an arc in $A \cup \{b_1\}$ having b_1 as an endpoint such that $\alpha_1 - \{b_1\} \subset A$, then by Theorem 1.4, if $x,y \in A-\alpha_1$, then $J \cup \alpha_1$ fails to separate x from y.

Now assume $b_1, b_2, \ldots, b_k \in J$ and $A = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ is a collection of pairwise disjoint arcs in $A \cup J$ such that for each $i = 1, 2, \ldots, k$, b_i is an endpoint of α_i and $\alpha_i - \{b_i\} \subset A$. Suppose further that $J \cup (\bigcup_{i=1}^{k} \alpha_i)$ fails to separate any two points $x,y \in A - (\bigcup_{i=1}^{k} \alpha_i)$. Then if α_{k+1} is an arc in $A \cup J$ with endpoint $b_{k+1} \in J$ such that $\alpha_{k+1} - \{b_{k+1}\} \subset A$ and $\alpha_{k+1} \cap (\bigcup_{i=1}^{k} \alpha_i) = \phi$, then $J \cup (\bigcup_{i=1}^{k} \alpha_i) \cup \alpha_{k+1}$ fails to separate x from y by Theorem 1.4.

We will now show that such arcs exist.

LEMMA 3.17: Let $J \subset E^2$ be a Jordan curve, A one of its complementary domains, and L a polygon in A. (If $A = J_U$, then we will restrict L such that $J \subset L_B$.) Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be pairwise disjoint arcs in $A \cup J$ such that for each $i = 1,2,\ldots,k$, $\alpha_i \cap L = \{a_i\}$ and $\alpha_i \cap J = \{b_i\}$ where a_i and b_i are the endpoints of α_i, and α_i is locally polygonal modulo b_i. Then if $b \in J - \bigcup_{i=1}^{k} \alpha_i$, for some $a \in L$, there exists an arc α in $A \cup J$ such that $\alpha \cap (\bigcup_{i=1}^{k} \alpha_i) = \phi$, $\alpha \cap L = \{a\}$, and $\alpha \cap J = \{b\}$ where a and b are the endpoints of α, and α is locally polygonal modulo b.

PROOF: Let $b \in J - \bigcup_{i=1}^{k} \alpha_i$ and $x \in L - \bigcup_{i=1}^{k} \alpha_i$. By Lemma 3.14, there exists a 1-1 path $p: I \to A \cup \{b\}$ such that $p(0) = x$, $p[0,1) \subset A$, $p(1) = b$, and $p(I)$ is locally polygonal modulo b.

Since p is continuous, for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $z \in U(b,\delta) \cap p(I)$, the arc from z to b in $p(I)$, denoted zb, lies completely in $U(b,\epsilon)$. Let $\epsilon = d(b,(\bigcup_{i=1}^{k}\alpha_i) \cup L)$ and let $y \in U(b,\delta) \cap p[0,1)$. By Remark 3.16, $J \cup (\bigcup_{i=1}^{k}\alpha_i)$ fails to separate x from y. Thus, there exists a 1-1 polygonal path $q:I \to A$ such that $q(0) = x$, $q(1) = y$, and $q(I) \subset A - \bigcup_{i=1}^{k}\alpha_i$. Let s be the largest element of $q^{-1}(q(I) \cap L)$, and let w be the smallest element of $\{q^{-1}(q(I) \cap yb)\} \cap [s,1]$ where yb is the arc in $p(I)$ from y to b. Let $q(s) = a$, and let $q[s,w] \cup q(w)b = \alpha$ where $q(w)b$ is the arc in yb from $q(w)$ to b. By construction, α is an arc in $A \cup J$ satisfying the required conditions. □

<u>LEMMA 3.18</u>: Let $J \subset E^2$ be a Jordan curve, A one of its complementary domains, and L a polygon in A where if $A = J_U$, L is restricted such that $J \subset L_B$. If for $n \in Z^+$, $b_1, b_2, \ldots, b_n$ are distinct elements of J, then there exist pairwise disjoint arcs $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $A \cup J$ such that for each i, $\alpha_i \cap L = \{a_i\}$, $\alpha_i \cap J = \{b_i\}$ where a_i and b_i are the endpoints of α_i, and α_i is locally polygonal modulo b_i.

PROOF: The lemma follows by Lemmas 3.15 and 3.17 and induction. □

<u>DEFINITION 3.19</u>: Let A be a nonempty set in the metric space X, and let $\epsilon > 0$. The ϵ-<u>neighborhood of A</u> is defined to be $U(A,\epsilon) = \cup\{U(a,\epsilon) \mid a \in A\}$.

<u>DEFINITION 3.20</u>: Let A be an arc in E^2, and let $a,b \in A$, $a \neq b$. Fix a 1-1 map $p:I \to A$ such that $p(I) = A$. Then it will be said that

a is less than b, denoted $a < b$, if $p^{-1}(a) < p^{-1}(b)$.

REMARK 3.21: The reader will note that whether $a < b$ or $b < a$ in Definition 3.20 will depend upon how the 1-1 map p is defined (i.e. which endpoint of the arc $p(I)$ is the image of zero under p). This will usually be clear from context. Otherwise, the order of p will be established. Note also that throughout the remainder of this chapter, the arc from a to b in $p(I)$ will be denoted ab, indicating the order from a to b.

LEMMA 3.22: Let $K \subset E^2$ be a Jordan curve. Let p_1 and p_2 be distinct points of K such that for $i = 1, 2$, there exists a polygonal arc in K containing p_i as an interior point. Let A be an arc in K having p_1 and p_2 as endpoints, and let $\epsilon > 0$. Then there exists a polygonal arc B such that

1) B has endpoints p_1 and p_2 and $B - \{p_1, p_2\} \subset K_B$, and
2) $B \subset U(A,\epsilon)$.

PROOF: By Lemma 3.10, for each point $x \in A$, there exists $\delta_x > 0$ such that if $r,s \in U(x, \delta_x) \cap K_B$, then r and s may be joined by a polygonal path in $U(x,\epsilon) \cap K_B$. We may assume $\delta_x < \epsilon$. Let $W = \{U(x, \delta_x) | x \in A\}$. From Theorem 3.7, there exists $\lambda > 0$ such that if $c,d \in A$ and $|c-d| < \lambda$, then there exists a set in W containing both c and d. Let $p_1 = x_1 < x_2 < \dots < x_n = p_2$ be points in A which partition A into components of diameter less than λ. Now there exist points $y_1, y_2, \dots, y_{n-1}$ in A such that for each $i = 1, 2, \dots, n-1$,

x_i, $x_{i+1} \in U(y_i, \delta_{y_i})$. For each $i = 1, 2, \ldots, n-2$, let $z_i \in U(y_i, \delta_{y_i}) \cap U(y_{i+1}, \delta_{y_{i+1}}) \cap K_B$. (Note that $U(y_i, \delta_{y_i}) \cap U(y_{i+1}, \delta_{y_{i+1}}) \cap K_B \neq \phi$ since for $i = 1, 2, \ldots, n-2$, $x_{i+1} \in U(y_i, \delta_{y_i}) \cap U(y_{i+1}, \delta_{y_{i+1}})$ and $x_{i+1} \in Fr(K_B)$.)

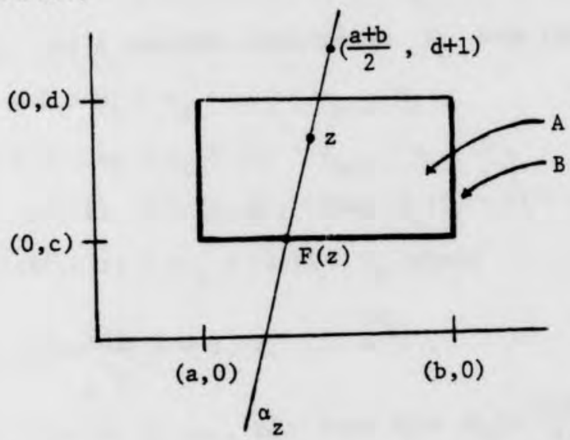
Since p_1 lies in the interior of a polygonal arc C in K, p_1 is either a vertex of C or lies in the interior of some line segment, say D, of C. In either case, there exists a line segment E lying in $(K \cup K_B) \cap U(y_1, \delta_{y_1})$ such that $E \cap K = \{p_1\}$. For if p_1 is a vertex, then $\{p_1\}$ is the intersection of two adjacent line segments of C, say G and H, and we may let E be a line segment of length less than μ lying in $K \cup K_B$ which bisects the angle determined by G and H where $\mu = \min\{\delta_{y_1} - |x_1 - y_1|, d(p_1, K - \text{Int}(G \cup H))\}$. On the other hand, if $p_1 \in \text{Int}(D)$, let $\gamma = \min\{\delta_{y_1} - |x_1 - y_1|, d(p_1, K - \text{Int}\,(D))\}$, and let E be a line segment perpendicular to D of length less than γ lying in $K \cup K_B$. Similarly, there exists a line segment F lying in $(K \cup K_B) \cap U(y_{n-1}, \delta_{y_{n-1}})$ such that $F \cap K = \{p_2\}$. Let z_0 and z_{n-1} be the other endpoints of E and F, respectively.

Now for each $i = 1, 2, \ldots, n-1$, z_{i-1} and z_i may be joined by a polygonal path in $U(y_i, \epsilon) \cap K_B$. Thus, by the gluing lemma, we have a polygonal path $f: I \to U(A, \epsilon)$ joining p_1 and p_2 such that $f(I) - \{p_1, p_2\} \subset K_B$. By Lemma 3.11, there exists a polygonal arc B such that B has endpoints p_1 and p_2, $B - \{p_1, p_2\} \subset K_B$, and $B \subset U(A, \epsilon)$. □

<u>3.23</u>: $\text{diam}(B) < \text{diam}(A) + 2\epsilon$. □

<u>LEMMA 3.24</u>: Let $a,b,c,d \in E^1$ such that $a < b$, $c < d$, and $b - a < 1$. Then there exists a map $H:e[a,b] \times [c,d] \to \{e(a)\} \times [c,d] \cup e[a,b] \times \{c\} \cup \{e(b)\} \times [c,d]$ such that $H(x) = x$ for all $x \in \{e(a)\} \times [c,d] \cup e[a,b] \times \{c\} \cup \{e(b)\} \times [c,d]$.

PROOF: Let $A = [a,b] \times [c,d]$ and $B = \{a\} \times [c,d] \cup [a,b] \times \{c\} \cup \{b\} \times [c,d]$. We regard A and B as being subsets of E^2 in the following. If $z \in A$. Let α_z denote the line containing z and $(\frac{a+b}{2}, d+1)$, and let $F:A \to B$ be defined by $B \cap \alpha_z = \{F(z)\}$. (See the figure below.)



We claim that F is continuous. To this end, let $\epsilon > 0$ and $z \in A$. Let $C = [a,b] \times \{d\}$, and let $D = [a,b] \times \{c\}$.

If $F(z) \in B-C$, we may assume $\epsilon < d(F(z),C)$ so that $Fr(U(F(z),\epsilon))$ intersects B exactly twice. Let $Fr(U(F(z),\epsilon)) \cap B = \{u,v\}$, and let $\delta_1 = d(z,\alpha_u)$, $\delta_2 = d(z,\alpha_v)$, and $\delta = \min\{\delta_1,\delta_2\}$. Now if $x \in U(z,\delta)$, then $F(x) \in U(F(z),\epsilon)$.

If $F(z) = (a,d)$, we may assume $\epsilon < d(F(z), D \cup (b,d))$ so that $Fr(U(F(z),\epsilon)$ intersects B exactly once. Let $Fr(U(F(z),\epsilon)) \cap B = \{y\}$, and let $\delta = d(z,\alpha_y)$. Now if $x \in U(z,\delta)$, then $F(x) \in U(F(z),\epsilon)$. Similarly, if $F(z) = (b,d)$, we may assume $\epsilon < d(F(z), D \cup (a,d))$, and the proof is essentially the same.

Now since $b - a < 1$, the map $G{:}A \to e[a,b] \times [c,d]$ defined by $G(r,s) = (e(r),s)$ for all $(r,s) \in A$, is a homeomorphism. Define the map $H{:}e[a,b] \times [c,d] \to \{e(a)\} \cup [c,d] \cup e[a,b] \times \{c\} \cup \{e(b)\} \times [c,d]$ by $H = G|B \circ F \circ G^{-1}$. $\square$

<u>LEMMA 3.25</u>: Let $J \subset E^2$ be a Jordan curve. Suppose that for $i \in \{1, 2, \ldots, k\}$, there exist a map $h_i{:}S^1 \times I \to J \cup J_B$, a positive integer m_i, and a positive real number μ_i such that

1) $m_0 = 2 \leq m_1 < m_2 < \ldots < m_{k-1} < m_k$,
2) $\mu_0 = 0 < \mu_1 < \mu_2 < \ldots < \mu_{k-1} < \mu_k < 1$,
3) if $i \in \{1, 2, \ldots, k\}$, then $h_i(C_i \times \{1\}) \subset J$ and $h_i[(S^1 \times I) - (C_i \times \{1\})] \subset J_B$ where

   $C_i = \{e(\frac{2\pi j}{2^{m_i}}) | j = 1, 2, \ldots, 2^{m_i}\}$,

4) if $i \in \{1, 2, \ldots, k\}$, then diam $(h_i[e[\frac{2\pi j}{2^{m_i}}, \frac{2\pi(j+1)}{2^{m_i}}] \times [\mu_i, 1]]) < \frac{1}{2^i}$

   for $j = 1, 2, \ldots, 2^{m_i}$,

5) if $i \in \{1, 2, \ldots, k\}$, then $h_i|[(S^1 \times [0,\mu_i]) \cup (C_i \times I)]$ is an embedding, $h_i(S^1 \times [0,\mu_i])$ is a polyhedran (i.e., the union of a finite collection of triangles), and for $j = 1,2,\ldots, 2^{m_i}$, $h_i(\{e(\frac{2\pi j}{2^{m_i}})\} \times I)$ is locally polygonal modulo

   $h_i(e(\frac{2\pi j}{2^{m_i}}), 1)$, and

6) if $i \in \{1, 2, \ldots, k\}$, then

$$h_{i+1}|[(S^1 \times [0,\mu_i]) \cup (C_i \times I)] = h_i|[(S^1 \times [0,\mu_i]) \cup (C_i \times I)].$$

Then there exist a map $h_{k+1}: S^1 \times I \to J \cup J_B$, a positive integer m_{k+1}, and a positive real number μ_{k+1} such that

1) $m_k < m_{k+1}$

2) $\mu_k < \mu_{k+1} < 1$

3) $h_{k+1}(C_{k+1} \times \{1\}) \subset J$ and $h_{k+1}[(S^1 \times I) - (C_{k+1} \times \{1\})] \subset J_B$

where $C_{k+1} = \{e(\frac{2\pi j}{2^{m_{k+1}}}) \mid j = 1, 2, \ldots, 2^{m_{k+1}}\}$,

4) $\operatorname{diam} (h_{k+1}[e[\frac{2\pi j}{2^{m_{k+1}}}, \frac{2\pi(j+1)}{2^{m_{k+1}}}] \times [\mu_{k+1}, 1]) < \frac{1}{2^{k+1}}$

for $j = 1, 2, \ldots, 2^{m_{k+1}}$,

5) $h_{k+1}|[(S^1 \times [0, \mu_{k+1}]) \cup (C_{k+1} \times I)]$ is an embedding,

$h_{k+1}(S^1 \times [0, \mu_{k+1}]$ is a polyhedran, and for $j = 1,2,\ldots,2^{m_{k+1}}$,

$h_{k+1}(\{e(\frac{2\pi j}{2^{m_{k+1}}})\} \times I)$ is locally polygonal modulo

$h_{k+1}(e(\frac{2\pi j}{2^{m_{k+1}}}), 1)$, and

6) $h_{k+1}|[(S^1 \times [0, \mu_k]) \cup (C_k \times I)] = h_k|[(S^1 \times [0, \mu_k]) \cup (C_k \times I)]$.

PROOF: Let $B = \{b_1, b_2, \ldots, b_{2^{m_{k+1}}}\}$, $m_{k+1} > m_k$, be a subset of J which contains $h_k(C_k \times \{1\})$ and partitions J into components of diameter less than $\frac{1}{7 \cdot 2^{k+1}}$. We assume that the members of B have been labeled consecutively around J. Let $L = h_k(S^1 \times \{\mu_k\})$. Let

$A = \{\alpha_1, \alpha_2, \ldots, \alpha_{2^{m_{k+1}}}\}$ be a collection of pairwise disjoint arcs containing $\{h_k(\{c\} \times [\mu_k, 1]) \mid c \in C_k\}$ such that for each $i = 1, 2, \ldots, 2^{m_{k+1}}$, $\alpha_i \cap L = \{a_i\}$ and $\alpha_i \cap J = \{b_i\}$ where a_i and b_i are the endpoints of α_i, and α_i is locally polygonal modulo b_i. If $i \in \{1, 2, \ldots, 2^{m_{k+1}}\}$, there exists $q_i \in \text{Int}(\alpha_i)$ such that the subarc β_i of α_i from q_i to b_i has diameter less than $\frac{1}{7\cdot 2^{k+1}}$. We may further assume that there exists a real number $\mu_{k+1} < 1$ such that if $b_j = h_k(c,1)$ for some $c \in C_k$, then $q_j = h_k(c,\mu_{k+1})$. For $i \in \{1, 2, \ldots, 2^{m_{k+1}}\}$, let γ_i be the arc of L from a_i to $a_{(i+1)\bmod 2^{m_{k+1}}}$ whose interior fails to intersect $\bigcup_{i=1}^{2^{m_{k+1}}} a_i$, and let θ_i be the arc of J from b_i to $b_{(i+1)\bmod 2^{m_{k+1}}}$, whose interior fails to intersect B. If $i \in \{1, 2, \ldots, 2^{m_{k+1}}\}$, let $K_i = \gamma_i \cup \alpha_i \cup \theta_i \cup \alpha_{(i+1)\bmod 2^{m_{k+1}}}$, and let $A_i = \beta_i \cup \theta_i \cup \beta_{(i+1)\bmod 2^{m_{k+1}}}$. We apply Lemma 3.22 to obtain a polygonal arc B_i joining q_i and $q_{(i+1)\bmod 2^{m_{k+1}}}$ such that $\text{Int}(B_i) \subset (K_i)_B$ and $B_i \subset U(A_i, \frac{1}{7\cdot 2^{k+1}})$. Note that by 3.23, diam $B_i <$ diam $A_i + \frac{2}{7\cdot 2^{k+1}}$; but diam $A_i < \frac{3}{7\cdot 2^{k+1}}$, so diam $B_i < \frac{5}{7\cdot 2^{k+1}}$. Hence diam $(\beta_i \cup B_i \cup \beta_{(i+1)\bmod 2^{m_{k+1}}}) < \frac{1}{2^{k+1}}$.

Now for $i = 1, 2, \ldots, 2^{m_{k+1}}$, let f_i be a homeomorphism of $\{e(\frac{2\pi i}{2^{m_{k+1}}})\} \times [\mu_k, 1]$ onto α_i such that $f_i(e(\frac{2\pi i}{2^{m_{k+1}}}), \mu_k) = a_i$, $f_i(e(\frac{2\pi i}{2^{m_{k+1}}}), \mu_{k+1}) = q_i$ and $f_i = h_k|\{c\} \times [\mu_k, 1]$ if $e(\frac{2\pi i}{2^{m_{k+1}}}) = c \in C_k$; let $g_i = h_k|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times \{\mu_k\}$; and let p_i be a homeomorphism of $e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times \{\mu_{k+1}\}$ onto B_i such that $p_i(e(\frac{2\pi i}{2^{m_{k+1}}}), \mu_{k+1}) = q_i$ and $p_i(e(\frac{2\pi(i+1)}{2^{m_{k+1}}}), \mu_{k+1}) = q_{(i+1) \bmod 2^{m_{k+1}}}$.

By Theorem 1.12, for each $i = 1, 2, \ldots, 2^{m_{k+1}}$, there exists a homeomorphism H_i of $e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times [\mu_k, \mu_{k+1}]$ onto $F_i = J_i \cup (J_i)_B$, where $J_i = \gamma_i \cup a_i q_i \cup B_i \cup q_{(i+1) \bmod 2^{m_{k+1}}} a_{(i+1) \bmod 2^{m_{k+1}}}$, such that

1) $H_i|\{e(\frac{2\pi i}{2^{m_{k+1}}})\} \times [\mu_k, \mu_{k+1}] = f_i|\{e(\frac{2\pi i}{2^{m_{k+1}}})\} \times [\mu_k, \mu_{k+1}]$,

2) $H_i|\{e(\frac{2\pi(i+1)}{2^{m_{k+1}}}) \times [\mu_k, \mu_{k+1}] = f_{(i+1) \bmod 2^{m_{k+1}}}|\{e(\frac{2\pi(i+1)}{2^{m_{k+1}}})\} \times [\mu_k, \mu_{k+1}]$,

3) $H_i|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times \{\mu_k\} = g_i$, and

4) $H_i|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times \{\mu_{k+1}\} = p_i|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times \{\mu_{k+1}\}$.

Now by the gluing lemma, the map $h'_{k+1}:[S^1 \times [\mu_k,\mu_{k+1}]) \cup (C_{k+1} \times [\mu_{k+1},1])] \to [(\bigcup_{i=1}^{2^{m_{k+1}}} F_i) \cup A]$, defined by

1) $h'_{k+1}|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times [\mu_k,\mu_{k+1}] = H_i|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times [\mu_k,\mu_{k+1}]$, and

2) $h'_{k+1}|C_{k+1} \times [\mu_{k+1},1] = f_i|C_{k+1} \times [\mu_{k+1},1]$,

is a homeomorphism, where $C_{k+1} = \{e(\frac{2\pi j}{2^{m_{k+1}}}) \mid j = 1, 2, \ldots, 2^{m_{k+1}}\}$.

For $i = 1, 2, \ldots, 2^{m_{k+1}}$, let

$$W_i=\{(\{e(\frac{2\pi i}{2^{m_{k+1}}})\}\times[\mu_{k+1},1])\cup(e[\frac{2\pi i}{2^{m_{k+1}}},\frac{2\pi(i+1)}{2^{m_{k+1}}}]\times\{\mu_{k+1}\})\cup(\{e(\frac{2\pi(i+1)}{2^{m_{k+1}}})\}\times[\mu_{k+1},1])\}.$$

By Lemma 3.24, for each $i = 1, 2, \ldots, 2^{m_{k+1}}$, there exists a map $n_i:e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times [\mu_{k+1},1] \to W_i$ such that $n_i(x) = x$ for all $x \in W_i$. Let the map $o_i:W_i \to \beta_i \cup B_i \cup \beta_{(i+1)\bmod 2^{m_{k+1}}}$ be defined such that $o_i(e(\frac{2\pi i}{2^{m_{k+1}}}),1) = b_i$ and $o_i(e(\frac{2\pi(i+1)}{2^{m_{k+1}}}),1) = b_{(i+1)\bmod 2^{m_{k+1}}}$.

Define the map $G_i:e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times [\mu_{k+1},1] \to \beta_i \cup B_i \cup \beta_{(i+1)\bmod 2^{m_{k+1}}}$ by $G_i = o_i \circ n_i$.

By the gluing lemma, we have a map $h''_{k+1}: S^1 \times [\mu_{k+1}, 1] \to$ $\left(\bigcup_{i=1}^{2^{m_{k+1}}} (\beta_i \cup B_i \cup \beta_{(i+1) \bmod 2^{m_{k+1}}})\right)$ defined by $h''_{k+1}|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times [\mu_{k+1}, 1] = G_i|e[\frac{2\pi i}{2^{m_{k+1}}}, \frac{2\pi(i+1)}{2^{m_{k+1}}}] \times [\mu_{k+1}, 1]$.

Now since $h''_{k+1}|(S^1 \times \{\mu_{k+1}\}) = h'_{k+1}|(S^1 \times \{\mu_{k+1}\})$ and $h'_{k+1}|(S^1 \times \{\mu_k\}) \cup (C_k \times [\mu_k, 1]) = h_k|(S^1 \times \{\mu_k\}) \cup (C_k \times [\mu_k, 1])$, the function $h_{k+1}: S^1 \times I \to J \cup J_B$ defined by

$$h_{k+1}(x) = \begin{cases} h_k(x) & \text{if } x \in (S^1 \times [0, \mu_k]) \cup (C_k \times I) \\ h'_{k+1}(x) & \text{if } x \in (S^1 \times [\mu_k, \mu_{k+1}]) \cup (C_{k+1} \times [\mu_{k+1}, 1]) \\ h''_{k+1}(x) & \text{if } x \in (S^1 \times [\mu_{k+1}, 1]) \end{cases}$$

is continuous. $\square$

<u>REMARK 3.26</u>: In the proof of the existence of an $h_1: S^1 \times I \to J \cup J_B$, a positive integer m_1, and a positive real number μ_1, which all satisfy our inductive hypothesis in Lemma 3.25, compatibility (property 6) is not applicable. Otherwise, the proof is identical to the proof of Lemma 3.25 and will therefore be omitted. Now by this remark, Lemma 3.25, and induction, the following theorem holds.

<u>THEOREM 3.27</u>: Let $J \subset E^2$ be a Jordan curve. For $n \in Z^+$, there exist a map $h_n: S^1 \times I \to J \cup J_B$, a positive integer m_n, and a real number μ_n such that the following properties hold:

1) $m_0 = 2 \leq m_1 < m_2 < \dots < m_{n-1} < m_n$

2) $\mu_0 = 0 < \mu_1 < \mu_2 < \dots < \mu_{n-1} < \mu_n < 1,$

3) $h_n(C_n \times \{1\}) \subset J$ and $h_n[(S^1 \times I) - (C_n \times \{1\})] \subset J_B$

where $C_n = \{e(\frac{2\pi j}{2^{m_n}}) \mid j = 1, 2, \dots, 2^{m_n}\}$,

4) $\operatorname{diam}(h_n[e[\frac{2\pi j}{2^{m_n}}, \frac{2\pi(j+1)}{2^{m_n}}] \times [\mu_n, 1]]) < \frac{1}{2^n}$

for $j = 1, 2, \dots, 2^{m_n}$,

5) $h_n|[(S^1 \times [0,\mu_n]) \cup (C_n \times I)]$ is an embedding,

$h_n(S^1 \times [0,\mu_n]$ is a polyhedran, and for

$j = 1, 2, \dots, 2^{m_n}$, $h_n(\{e(\frac{2\pi j}{2^{m_n}})\} \times I)$ is locally

polygonal modulo $h_n(e(\frac{2\pi j}{2^{m_n}}), 1)$, and

6) $h_{n+1}|[(S^1 \times [0,\mu_n]) \cup (C_n \times I)] = h_n|[(S^1 \times [0,\mu_n]) \cup (C_n \times I)]$.□

REMARK 3.28: Following a number of preliminary lemmas, we shall show that the sequence of functions $\{h_i\}_{i=1}^{\infty}$, given by Theorem 3.27, converges uniformly on $S^1 \times I$.

LEMMA 3.29: If $\epsilon > 0$, there exists a positive integer i such that $h_i(C_i \times \{1\})$ partitions J into components of diameter less than ϵ .

PROOF: Suppose not. Then there exists a chain $P_1 \supset P_2 \supset \ldots$ such that P_j is a component of $J - h_j(C_j \times \{1\})$ and diam $P_j \geq \epsilon$. Let x and y be the endpoints of $\bigcap_{j=1}^{\infty} P_j$. Then $|x - y| \geq \epsilon$. Choose k so that $\frac{1}{2^k} < \frac{1}{3}|x - y|$. There exist positive integers $n \geq k$ and q such that $|x - h_n(e(\frac{2\pi q}{2^{m_n}}),1)| < \frac{1}{3}|x - y|$ and $|y - h_n(e(\frac{2\pi(q\pm 1)}{2^{m_n}}), 1|$ $< \frac{1}{3}|x - y|$. Then

$$|x-y| \leq |x-h_n(e(\frac{2\pi q}{2^{m_n}}),1)| + |h_n(e(\frac{2\pi q}{2^{m_n}}),1)-h_n(e(\frac{2\pi(q\pm 1)}{2^{m_n}}),1)| + |h_n(e(\frac{2\pi(q\pm 1)}{2^{m_n}}),1)-y|$$

$$< \frac{1}{3}|x - y| + \frac{1}{2^n} + \frac{1}{3}|x - y| < |x - y| .$$

This is a contradiction. Therefore, the lemma holds. □

<u>REMARK 3.30</u>: Due to Lemma 3.29, we may suppose, relabeling if necessary, that $h_i(C_i \times \{1\})$ partitions J into components of diameter less than $\frac{1}{2^i}$.

<u>LEMMA 3.31</u>: Let $J \subset E^2$ be a Jordan curve such that $J = A \cup B$ where A and B are arcs which intersect only at their endpoints. If diam $A = a$ and diam $B = b$, then diam $J \leq a + b$.

PROOF: Let $A \cap B = \{c,d\}$. If $x \in A$ and $y \in B$, then $|x - y| \leq |x - c| + |c - y| \leq a + b$. Therefore diam $J \leq a + b$. □

<u>REMARK 3.32</u>: Let $k \in Z^+$. For each $j = 1, 2, \ldots, 2^{m_k}$, let
$A_{kj} = \{(\{e(\frac{2\pi j}{2^{m_k}})\} \times [\mu_k,1]) \cup (e[\frac{2\pi j}{2^{m_k}}, \frac{2\pi(j+1)}{2^{m_k}}] \times \{\mu_k\}) \cup (\{e(\frac{2\pi(j+1)}{2^{m_k}})\} \times [\mu_k,1])\}$,

and let D_{kj} be the Jordan curve such that $D_{kj} = h_k(A_{kj}) \cup \Theta_k$ where Θ_k is the arc in J from $h_k(e(\frac{2\pi j}{2^{m_k}}),1)$ to $h_k(e(\frac{2\pi(j+1)}{2^{m_k}}), 1)$ whose interior fails to intersect $h_k(C_k \times \{1\})$. The reader should note that this notation will be used throughout the remainder of this chapter.

By Theorem 3.27, diam $(h_k[e[\frac{2\pi j}{2^{m_k}}, \frac{2\pi(j+1)}{2^{m_k}}] \times [\mu_k,1]]) < \frac{1}{2^k}$ for $j = 1, 2, \ldots, 2^{m_k}$. By Lemma 3.29 and Remark 3.30, diam $\Theta_k < \frac{1}{2^k}$. Thus by Lemma 3.31, diam $D_{kj} < \frac{1}{2^{k-1}}$.

<u>LEMMA 3.33</u>: Let $k \in Z^+$. For all $n > k$, $h_n(S^1 \times [0,\mu_k]) \cap (D_{kj})_B = \phi$.

PROOF: Suppose $h_n(S^1 \times [0,\mu_k]) \cap (D_{kj})_B \neq \phi$. Then since $h_n|[(S^1 \times [0,\mu_k]) \cup A_{kj}]$ is an embedding, $h_n(S^1 \times [0,\mu_k]) \subset D_{kj} \cup D_{kj})_B$, and $h_n\{[C_k \times \{\mu_k\}] - \{e(\frac{2\pi j}{2^{m_k}}), e(\frac{2\pi(j+1)}{2^{m_k}})\}\} \subset (D_{kj})_B$. Now since $h_n|[C_k \times I) \cup (S^1 \times [0,\mu_k])]$ is an embedding, the arc $h_n(\{e(\frac{j-1}{2^{m_k}})\} \times I)$ fails to intersect $h_n(A_{kj})$. Thus, by the Jordan curve theorem, $h_n(e(\frac{2\pi(j-1)}{2^{m_k}}), 1) \in \Theta_k$. This is a contradiction. Therefore the lemma holds. □

<u>REMARK 3.34</u>: Now from Lemma 3.33, if $k \in Z^+$, then for all $n > k$, there exists a positive integer p such that $h_n(\{e(\frac{2\pi p}{2^{m_n}})\} \times [\mu_k,1]) \cap (D_{kj})_B \neq \phi$.

<u>LEMMA 3.35</u>: Let $k \in Z^+$. For any integer $n > k$ and any positive integer p such that $h_n(e(\frac{2\pi p}{2^{m_n}}) \times [\mu_k, 1]) \cap (D_{kj})_B \neq \phi$, $h_n(e(\frac{2\pi p}{2^{m_n}}) \times [\mu_k, 1]) \subset (D_{kj})_B \cup (D_{kj})_B$ and intersects D_{kj} at exactly two points (one of which lies in $h_k(A_{kj})$ and the other in $J \cap D_{kj}$).

PROOF: Denote $e(\frac{2\pi p}{2^{m_n}}) \times [\mu_k, 1]$ by Q. Since $h_n|[(S^1 \times [0, \mu_n]) \cup (C_n \times I)$ is an embedding, $h_n|(A_{kj} \cup Q)$ is an embedding. Now $A_{kj} \cap Q = (e(\frac{2\pi p}{2^{m_n}}), \mu_k)$ and $a = h_n(e(\frac{2\pi p}{2^{m_n}}), \mu_k) \in h_k(A_{kj})$. Also $Q \cap (C_n \times \{1\}) = (e(\frac{2\pi p}{2^{m_n}}), 1)$ and $b = h_n(e(\frac{2\pi p}{2^{m_n}}), 1) \in J$. Since $h_n(Q)$ is an arc and $h_n[(S^1 \times I) - (C_n \times \{1\})] \subset J_B$, $h_n(Q)$ intersects J at exactly one point, namely b. Now if $b \in J - D_{kj}$, $h_n(Q)$ intersects $h_n(A_{kj}) - \{a\}$. This is impossible; so $b \in J \cap D_{kj}$. Furthermore, $h_n(Q) \subset D_{kj} \cup (D_{kj})_B$ since $h_n(Q) \cap D_{kj} = \{a,b\}$ and $h_n(Q) \cap (D_{kj})_B \neq \phi$. $\square$

<u>THEOREM 3.36</u>: The sequence of functions $\{h_i\}_{i=1}^{\infty}$ converges uniformly on $S^1 \times I$.

PROOF: Let $\epsilon < 0$. By the Archimedean principle, there exists a positive integer k such that $\frac{7}{2^k} < \epsilon$. Let p and q be integers greater than or equal to k. We may assume $p < q$.

If $x \in [(S^1 \times [0,\mu_p]) \cup (C_p \times I)] = X$, then $h_p(x) = h_q(x)$ and $|h_p(x) - h_q(x)| = 0 < \epsilon$.

Let $x \in [(S^1 \times [\mu_p,\mu_q]) - X]$. Choose a positive integer j such that $x \in \{e[\frac{2\pi j}{2^{m_p}}, \frac{2\pi(j+1)}{2^{m_p}}] \times [\mu_p, 1] = F$. There exists a positive integer r such that $x \in \{e[\frac{2\pi r}{2^{m_q}}, \frac{2\pi(r+1)}{2^{m_q}}] \times [\mu_p, 1]\}$. Let

$A = \{e(\frac{2\pi r}{2^{m_q}})\} \times [\mu_p,\mu_q]$, $B = \{e(\frac{2\pi(r+1)}{2^{m_q}})\} \times [\mu_p,\mu_q]$,

$C = e[\frac{2\pi r}{2^{m_q}}, \frac{2\pi(r+1)}{2^{m_q}}] \times \{\mu_p\}$, $D = e[\frac{2\pi r}{2^{m_q}}, \frac{2\pi(r+1)}{2^{m_q}}] \times \{\mu_q\}$, and

$E = e[\frac{2\pi r}{2^{m_q}}, \frac{2\pi(r+1)}{2^{m_q}}] \times [\mu_p,\mu_q]$. From Lemma 3.35, $h_q(A)$ and $h_q(B)$ are subsets of $(D_{pj})_B$, and therefore each has diameter less than $\frac{1}{2^{p-1}}$.

Since $h_q|C = h_p|C$, diam $h_q(C) < \frac{1}{2^p}$. Since $D \subset e[\frac{2\pi r}{2^{m_q}}, \frac{2\pi r}{2^{m_q}}] \times [\mu_q, 1]$, diam $h_q(D) < \frac{1}{2^q}$. Since $h_q|E$ is an embedding, $Fr(h_q(E))$ is a Jordan curve and $h_q(x) \in h_q(E)$. Furthermore, by the Lemma 3.31 and the Jordan curve theorem,

$$\begin{aligned}\mathrm{diam}(h_q(E)) &\le \mathrm{diam}(h_q(A) + \mathrm{diam}(h_q(B)) + \mathrm{diam}(h_q(C)) + \mathrm{diam}(h_q(D)) \\ &< \frac{1}{2^{p-1}} + \frac{1}{2^{p-1}} + \frac{1}{2^p} + \frac{1}{2^q} \\ &< \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}} + \frac{1}{2^k} + \frac{1}{2^k} = \frac{3}{2^{k-1}} .\end{aligned}$$

Now by property 4 of Theorem 3.27, $\operatorname{diam}(h_p(F)) < \frac{1}{2^p} \leq \frac{1}{2^k}$. By property 6, $h_p(F) \cap h_q(E) \neq \phi$. Therefore, since $h_p(x) \in h_p(F)$,

$|h_p(x) - h_q(x)| < \frac{3}{2^{k-1}} + \frac{1}{2^k} = \frac{7}{2^k} < \epsilon$.

Let $x \in (S^1 \times [\mu_q, 1] - X)$. Then $x \in F$ and $x \in \{e[\frac{2\pi r}{2^{m_q}}, \frac{2\pi(r+1)}{2^{m_q}}] \times [\mu_q, 1]\} = G$. There exists a positive integer s such that $x \in \{e[\frac{2\pi s}{2^{m_k}}, \frac{2\pi(s+1)}{2^{m_k}}] \times [\mu_k, 1]\}$. Now since $\operatorname{diam}(h_p(F)) < \frac{1}{2^p}$, $\operatorname{diam}(h_q(G)) < \frac{1}{2^q}$, and $h_p(F) \cap D_{ks} \neq \phi \neq D_{ks} \cap h_q(G)$, we have

$$
\begin{aligned}
|h_p(x) - h_q(x)| &< \frac{1}{2^p} + \frac{1}{2^{k-1}} + \frac{1}{2^q} \\
&< \frac{1}{2^k} + \frac{1}{2^{k-1}} + \frac{1}{2^k} \\
&= \frac{1}{2^{k-2}} < \frac{7}{2^k} < \epsilon \ .
\end{aligned}
$$

Therefore, $\{h_i\}_{i=1}^{\infty}$ converges uniformly on $S^1 \times I$. □

<u>REMARK 3.37</u>: Let $h: S^1 \times I \to J \cup J_B$ be defined by $\lim_{i\to\infty} h_i = h$. Since $\{h_i\}_{i=1}^{\infty}$ converges uniformly on $S^1 \times I$, h is continuous. In order to show that h is an embedding, we now show that h is 1-1. This will follow from the next lemma and a remark.

LEMMA 3.38: Let $b \in (S^1 \times \{1\}) - (\bigcup_{i=1}^{\infty} C_i \times \{1\})$. If k and j are positive integers such that $b \in e[\frac{2\pi j}{2^{m_k}}, \frac{2\pi(j+1)}{2^{m_k}}] \times \{1\}$, then $h(b) \in D_{kj} \cap J$.

PROOF: Since $\bigcup_{i=1}^{\infty} C_i \times \{1\}$ is dense in $S^1 \times \{1\}$, there exists a sequence $\{x_i\}$ of points of $(\bigcup_{i=1}^{\infty} C_i \times \{1\}) \cap (e[\frac{2\pi j}{2^{m_k}}, \frac{2\pi(j+1)}{2^{m_k}}] \times \{1\})$ converging to b where $x_i \in C_{j_i} \times \{1\}$. (Note that $k \leq j_i$ for all i.) By Lemma 3.35, $h(x_i) = h_{j_i}(x) \in J \cap D_{kj}$ for each i. Therefore since h is continuous and $J \cap D_{kj}$ is closed, $\{h(x_i)\} \to h(b) \in J \cap D_{kj}$. □

REMARK 3.39: From properties 3 and 6 of Theorem 3.27, if $b \in \bigcup_{i=1}^{\infty} C_i \times \{1\}$, then $h(b) \in J$. By Lemma 3.38, if $b \in (S^1 \times \{1\}) - (\bigcup_{i=1}^{\infty} C_i \times \{1\})$, then $h(b) \in J$. Therefore $h(S^1 \times \{1\}) \subset J$. Furthermore, from properties 3 and 6, $h(S^1 \times [0,1)) \subset J_B$.

THEOREM 3.40: The continuous function $h: S^1 \times I \to J \cup J_B$ is 1-1.

PROOF: We have four cases.

1) Let $a,b \in S^1 \times [0,1) \cup (\bigcup_{i=1}^{\infty} C_i \times I)$, $a \neq b$. Then there exists a positive integer k such that $a,b \in [(S^1 \times [0,\mu_k]) \cup (C_k \times I)]$. Since for all $n \geq k$, $h_n|[(S^1 \times [0,\mu_k]) \cup (C_k \times I)] = h_k|[(S^1 \times [0,\mu_k]) \cup (C_k \times I)]$ and $h_k|[(S^1 \times [0,\mu_k]) \cup (C_k \times I)]$ is an embedding, $h(a) = h_n(a) \neq h_n(b) = h(b)$.

2) Let $a \in S^1 \times [0,1)$ and $b \in (S^1 \times \{1\}) - (\bigcup_{i=1}^{\infty} C_i \times \{1\})$. There exists a positive integer k such that $a \in S^1 \times [0,\mu_k]$ and $h_k(a) \in J_B$

by Remark 3.39. Since for any $n \geq k$, $h_n(a) = h_k(a)$, we have $h(a) \in J_B$. By Lemma 3.38, $h(b) \in J$. Therefore, $h(a) \neq h(b)$.

3) Let $a \in \bigcup_{i=1}^{\infty} C_i \times \{1\}$ and $b \in (S^1 \times \{1\}) - (\bigcup_{i=1}^{\infty} C_i \times \{1\})$. There exist integers k and r such that $b \in (e[\frac{2\pi r}{2^{m_k}}, \frac{2\pi(r+1)}{2^{m_k}}] \times \{1\} = A$, and $a \notin A$ since $\bigcup_{i=1}^{\infty} C_i \times \{1\}$ is dense in $S^1 \times \{1\}$. By Lemma 3.38, $h(b) \in D_{kr} \cap J$, but by the definition of D_{kr}, $h(a) \notin D_{kr} \cap J$. Therefore $h(a) \neq h(b)$.

4) Let $a,b \in (S^1 \times \{1\}) - (\bigcup_{i=1}^{\infty} C_i \times \{1\})$, $a \neq b$. There exist positive integers k, r, and s such that $a \in (e[\frac{2\pi r}{2^{m_k}}, \frac{2\pi(r+1)}{2^{m_k}}]\times\{1\}) = A$ and $b \in (e[\frac{2\pi s}{2^{m_k}}, \frac{2\pi s}{2^{m_k}}] \times \{1\}) = B$ where $A \cap B = \phi$. By Lemma 3.38, $h(a) \in D_{kr} \cap J$ and $h(b) \in D_{ks} \cap J$. Since $(D_{kr} \cap J) \cap (D_{ks} \cap J) = \phi$, $h(a) \neq h(b)$. $\square$

<u>REMARK 3.41</u>: Let h be as above. Just as h was constructed, we may construct a homeomorphic embedding $g:S^1 \times [1,2] \to J \cup J_U$ such that $g(S^1 \times \{1\}) = J$. Let $\Pi:S^1 \times [1,2] \to S^1$ be the projection, and define the homeomorphism $k:S^1 \times [1,2] \to S^1 \times [1,2]$ by $k(x,t) = (\Pi \circ g^{-1}(h(x,1)),t)$ for all $x \in S^1$ and $t \in [1,2]$. Let $f = g \circ k:S^1 \times [1,2] \to J \cup J_U$. Note that for all $x \in S^1$, $f(x,1) = g(k(x,1)) = g(\Pi \circ g^{-1}(h(x,1)),1) = h(x,1)$. By applying the gluing lemma to the maps h and f and reparameterizing, we obtain the following.

THEOREM 3.42: If J is a Jordan curve in E^2, then there exists a homeomorphic embedding $h':S^1 \times I \to E^2$ such that $h'(S^1 \times \{\frac{1}{2}\}) = J$. $\square$

SUMMARY

From Chapter III, for any Jordan curve J in E^2, we have a homeomorphic embedding f of $S^1 \times I$ into E^2 such that $f(S^1 \times \{\frac{1}{2}\}) = J$. Let n be the "north pole" in S^2 and $G:S^2 - \{n\} \to E^2$ be the homeomorphism defined by stereographic projection. By Brown's generalized Schoenflies Theorem, the closure of either complementary domain of $G^{-1}(J)$ is a 2-cell. It is thus easy to see that there is a homeomorphism $g:S^2 \to S^2$ such that $g(G^{-1}(J))$ is the equator. We may further require $g(n) = n$; then if $h:E^2 \to E^2$ is the homeomorphism defined by $h(x) = G(g(G^{-1}(x)))$ for all $x \in E^2$, h carries J onto a circle in E^2. Thus we have established the classical Schoenflies Theorem. By the technique used to prove Theorem 1.12, we immediately obtain the following.

<u>THEOREM</u>: If J and K are Jordan curves in E^2 and $h:J \to K$ is a homeomorphism, then there exists a homeomorphism $h^*:E^2 \to E^2$ such that $h^*(x) = h(x)$ for all $x \in J$.

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